Scale locality and the inertial range in compressible turbulence

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Abstract

We use a coarse-graining approach to prove that inter-scale transfer of kinetic energy in compressible turbulence is dominated by local interactions. Locality here means that interactions between disparate scales decay at least as fast as a power-law function of the scale-disparity ratio. In particular, our results preclude transfer of kinetic energy from large-scales directly to dissipation scales, such as into shocks, in the limit of high Reynolds number turbulence as is commonly believed. The results hold in broad generality, at any Mach number, for any equation of state, and without the requirement of homogeneity or isotropy. The assumptions we make in our proofs on the scaling of velocity, pressure, and density structure functions are weak and enjoy compelling empirical support. Under a stronger assumption on pressure dilatation co-spectrum, we show that *mean* kinetic and internal energy budgets statistically decouple beyond a transitional "conversion" range. Our analysis demonstrates the existence of an ensuing inertial scale-range over which mean SGS kinetic energy flux becomes constant, independent of scale. Over this inertial range, mean kinetic energy cascades locally and in a conservative fashion, despite not being an invariant.

Key Words: Compressible turbulence; Scale locality; Cascade; Inertial range

1 Introduction

This paper is the second in a series which investigates the physical nature of compressible turbulence. In the first paper [1], we laid a framework to study the coupling of scales in such flows and to analyze transfer of kinetic energy between different scales. The aim here is to explore whether this transfer of energy takes place through a scale-local cascade process, similar to incompressible turbulence. This is of central importance in the subject because scale locality is necessary to warrant the concept of an inertial range and to justify the existence of universal statistics of turbulent fluctuations.

The traditional Richardson-Kolmogorov-Onsager picture of incompressible turbulence makes the fundamental assumption of a scale-local cascade process in which modes all of a comparable scale $\sim \ell$ (differing at most by some fixed ratio, typically of order 2) participate predominantly in the transfer of energy across scale ℓ. This also implies that energy transfer is primarily between modes at comparable scales, with a ratio of order 2. If, furthermore, the cascade steps are chaotic processes then it is expected that any "memory" of large-scale particulars of the system, such as geometry and large-scale statistics, or the specifics of microscopic dissipation, will be "forgotten." This gives rise to an *inertial scale-range* over which turbulent fluctuations have universal statistics and the flow evolves under its own internal dynamics without *direct* communication with the largest or smallest scales in the system.

Therefore, scale locality is crucial to justify the existence of an inertial range and its universal statistics, and is necessary for the physical foundation of large-eddy simulation (LES) modelling of turbulence. It motivates the belief that models of subscale terms in the equations for large-scales can be of general utility, independent of the particulars of turbulent flows under study.

[29, 30, 31] was the first to demonstrate locality in incompressible Navier-Stokes turbulence using detailed closure calculations. He showed that interactions between widely separated scales, $\ell_1 \ll \ell_2$, decay as a power-law of their ratio, $(\ell_1/\ell_2)^\alpha$, where $\alpha > 0$. Later on, [19] was able to prove locality rigorously from the equations of motion and under very mild assumptions, without any closure or statistical averaging. More recently, [2] proved locality using Fourier analysis and in [3], they showed that it also holds in incompressible magnetohydrodynamic turbulence. Furthermore, there has been several recent studies by []Domaradzki07a,Domaradzki09,AluieEyink09,AluieEyink10,Domaradzki10 which support the aforementioned theoretical results from direct numerical simulation (DNS) data.

No similar results, either theoretical or empirical, exist for compressible turbulence. The idea of a cascade itself is without physical basis since kinetic energy is not a global invariant of the inviscid dynamics. The notion of an inertial cascade-range in such flows is, therefore, still tenuous and unsubstantiated. In this paper, we shall prove rigorously under modest assumptions that inter-scale transfer of kinetic energy is indeed local in scale. Under a stronger assumption, we will further show that kinetic energy cascades conservatively despite not being an invariant.

The outline of this paper is as follows. In § 2 we present preliminary definitions and discussion. In the course of proving locality of the cascade, we shall first prove that the subgrid scale flux (defined below) is dominated by scale-local interactions. This is done by expressing the flux in terms of increments in § 3, then proving that increments themselves are scale-local in § 4, then, finally, showing how this leads to locality of the flux in § 5. In § 6, we discuss the implications on locality of the cascade itself. We show that beyond a transitional "conversion" range, an inertial range emerges over which kinetic energy cascades conservatively in a scale-local fashion. In § 7.1 and § 7.2, we argue for the validity of our assumptions based on empirical evidence and physical arguments. In § 7.4 and § 7.5 we discuss common misconceptions regarding scale locality in the presence of intermittency and shocks. We summarize our main results in § 8 and defer mathematical details to Appendices A and B.

2 Preliminaries

2.1 Governing dynamics

We prove locality of kinetic energy transfer by a direct analysis of the compressible Navier Stokes equations, without use of any closure approximation. The equations are those of continuity (1), momentum (2), and either internal energy (3) or total energy (4):

$$\partial_t\rho \quad +\partial_j(\rho u_j) = 0 \tag{1}$$

$$\partial_t(\rho u_i) \quad +\partial_j(\rho u_i u_j) = -\partial_i P + \mu\partial_j(\partial_j u_i + \frac{1}{3}\partial_m u_m \delta_{ij}) + \rho F_i \tag{2}$$

$$\partial_t(\rho e) \quad +\partial_j\left\{\rho e u_j - \mu(u_m\partial_m u_j - u_j\partial_m u_m)\right\} = -P\partial_j u_j + \mu|\partial_j u_i|^2 + \frac{\mu}{3}|\partial_j u_j|^2 - \partial_j q_j \tag{3}$$

$$\partial_t(\rho E) \quad +\partial_j(\rho E u_j) = -\partial_j(P u_j) + \mu\partial_j\{u_i[(\partial_j u_i + \partial_i u_j) - \frac{2}{3}\partial_m u_m \delta_{ij}]\} - \partial_j q_j + \rho u_i F_i \tag{4}$$

Here, $\mathbf{u}$ is velocity, ρ is density, e is internal energy per unit mass, $E = |\mathbf{u}|^2/2 + e$ is total energy per unit mass, P is pressure, μ is dynamic viscosity, $\mathbf{F}$ is an external acceleration field stirring the fluid, $\mathbf{q} = -\kappa\boldsymbol{\nabla}T$ is the heat flux with a conduction coefficient κ and temperature T. For convenience, we have assumed a zero bulk viscosity even though all our analysis applies to the more general case. We have also assumed that $\mu = \nu\rho$ is independent of $\mathbf{x}$.

2.2 Coarse-grained equations

Following [25] and [19], we presented in a previous paper, [1], a scale-decomposition based on coarse-graining which satisfies an *inviscid criterion*, *i.e.* it guarantees that viscous momentum diffusion and kinetic energy dissipation are negligible at large-scales. The decomposition yields a scale-range $L \gg \ell \gg$

ℓ_μ over which kinetic energy is immune from viscous dissipation and external injection by stirring. Here, L denotes "integral scale" and ℓ_μ denotes dissipation scale.

Using the coarse-graining approach, we can resolve dynamics *both in scale and in space*. We define a coarse-grained or (low-pass) filtered field in d-dimensions as

$$\overline{\mathbf{a}}_\ell(\mathbf{x}) = \int d^d\mathbf{r}\ G_\ell(\mathbf{r})\mathbf{a}(\mathbf{x}+\mathbf{r}), \tag{5}$$

where $G(\mathbf{r})$ is a smooth convolution kernel which decays sufficiently rapidly for large r, and is normalized, $\int d^d\mathbf{r}\ G(\mathbf{r}) = 1$. Its dilation $G_\ell(\mathbf{r}) \equiv \ell^{-d}G(\mathbf{r}/\ell)$ has its main support in a ball of radius ℓ. We also define a complimentary high-pass filter by

$$\mathbf{a}_\ell^{'}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) - \overline{\mathbf{a}}_\ell(\mathbf{x}). \tag{6}$$

In the rest of our paper, we shall take the liberty of dropping subscript ℓ whenever there is no risk of ambiguity.

In [1], we proved that viscous momentum diffusion and kinetic energy dissipation are negligible at large-scales when a Favre (or density-weighted) decomposition is employed. A Favre filtered field is weighted by density as

$$\tilde{f}_\ell(\mathbf{x}) \equiv \frac{\overline{\rho f}_\ell(\mathbf{x})}{\overline{\rho}_\ell(\mathbf{x})}. \tag{7}$$

The resultant large-scale dynamics for continuity and momentum are, respectively,

$$\partial_t\overline{\rho} + \partial_i(\overline{\rho}\tilde{u}_i) = 0. \tag{8}$$

$$\begin{aligned}\partial_t\overline{\rho}\tilde{u}_i + \partial_j(\overline{\rho}\tilde{u}_i\ \tilde{u}_j) &= -\partial_j\left(\overline{\rho}\ \tilde{\tau}(u_i,u_j)\right) - \partial_i\overline{P} \\ &+ \mu\partial_j\{[(\partial_j\overline{u}_i + \partial_i\overline{u}_j) - \frac{2}{3}\partial_m\overline{u}_m\delta_{ij}]\} + \overline{\rho}\widetilde{F}_i,\end{aligned} \tag{9}$$

where

$$\overline{\rho}\tilde{\tau}(u_i,u_j) \equiv \overline{\rho}(\widetilde{u_i u_j} - \tilde{u}_i\ \tilde{u}_i) \tag{10}$$

is the *turbulent stress* from the eliminated scales $< \ell$. It is also straightforward to derive a kinetic energy budget for the large-scale, which reads

$$\partial_t\overline{\rho}\frac{|\tilde{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla}\cdot\mathbf{J}_\ell = -\Pi_\ell - \Lambda_\ell + \overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell - D_\ell + \epsilon^{inj}, \tag{11}$$

where $\mathbf{J}_\ell(\mathbf{x})$ is space transport of large-scale kinetic energy, $\Pi_\ell(\mathbf{x}) + \Lambda_\ell(\mathbf{x})$, is the subgrid scale (SGS) kinetic energy flux to scales $< \ell$, $-\overline{P}\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}$ is large-scale pressure dilatation, $D_\ell(\mathbf{x})$ is viscous dissipation acting on scales $> \ell$, and $\epsilon^{inj}(\mathbf{x})$ is the energy injected due to external stirring. These terms are defined

as

$$\Pi_\ell(\mathbf{x}) = -\overline{\rho}\,\partial_j\tilde{u}_i\,\tilde{\tau}(u_i,u_j) \tag{12}$$

$$\Lambda_\ell(\mathbf{x}) = \frac{1}{\overline{\rho}}\partial_j\overline{P}\,\overline{\tau}(\rho,u_j) \tag{13}$$

$$D_\ell(\mathbf{x}) = \mu\left[\partial_j\tilde{u}_i\,\partial_j\overline{u}_i + \frac{1}{3}\partial_i\tilde{u}_i\,\partial_j\overline{u}_j\right] \tag{14}$$

$$J_j(\mathbf{x}) = \overline{\rho}\frac{|\tilde{\mathbf{u}}|^2}{2}\tilde{u}_j + \overline{P}\overline{u}_j + \tilde{u}_i\overline{\rho}\tilde{\tau}(u_i,u_j) - \mu\tilde{u}_i\partial_j\overline{u}_i - \frac{\mu}{3}\tilde{u}_j\partial_i\overline{u}_i \tag{15}$$

$$\epsilon^{inj}(\mathbf{x}) = \tilde{u}_i\,\overline{\rho}\widetilde{F}_i \tag{16}$$

where

$$\overline{\tau}_\ell(f,g) \equiv \overline{(fg)_\ell} - \overline{f}_\ell\overline{g}_\ell \tag{17}$$

in expression (13) is the 2^{nd}-order *generalized central moment* of fields $f(\mathbf{x}), g(\mathbf{x})$ (see [25]). Equations (8)-(11) describe the dynamics at scales $> \ell$, for arbitrary ℓ, at every point $\mathbf{x}$ and at every instant in time. They hold for each realization of the flow without any statistical averaging.

The SGS flux is comprised of deformation work, Π_ℓ, and baropycnal work, Λ_ℓ, which we discussed in some detail in [1]. These represent the only two processes capable of direct transfer of kinetic energy *across* scales. Pressure dilatation, $-\overline{P}_\ell\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell$, does not contain any modes at scales $< \ell$ (or a moderate multiple thereof), at least for a filter kernel $\hat{G}(\mathbf{k})$ compactly supported in Fourier space. Therefore, pressure dilatation cannot participate in the inter-scale transfer of kinetic energy and only contributes to conversion of large-scale kinetic energy into internal energy. This observation is one of the key ingredients to proving scale locality.

3 SGS flux in terms of increments

As was realized in the pioneering work of [19], there are two facts crucial for scale locality of the SGS flux across ℓ. First is that SGS flux can be written in terms of *increments*,

$$\delta\mathbf{a}(\mathbf{x};\mathbf{r}) = \mathbf{a}(\mathbf{x}+\mathbf{r}) - \mathbf{a}(\mathbf{x}), \tag{18}$$

for separation distances $|\mathbf{r}| < \ell$ (or some moderate multiple of ℓ) and do not depend on the absolute field $\mathbf{a}(\mathbf{x})$.

The SGS flux terms can be expressed in terms of increments by noting that gradient fields and central

moments are related to increments as[1],

$$\begin{aligned}
\nabla \overline{f}_\ell &= \mathcal{O}[\delta f(\ell)/\ell], && (19)\\
f'_\ell &= \mathcal{O}[\delta f(\ell)], && (20)\\
\overline{\tau}_\ell(f,g) &= \mathcal{O}[\delta f(\ell)\ \delta g(\ell)], && (21)\\
\nabla\overline{\tau}_\ell(f,g) &= \mathcal{O}[\delta f(\ell)\ \delta g(\ell)/\ell], && (22)\\
\overline{\tau}_\ell(f,g,h) &= \mathcal{O}[\delta f(\ell)\ \delta g(\ell)\ \delta h(\ell)], && (23)
\end{aligned}$$

where the symbol $\mathcal{O}$ stands for "same order-of-magnitude as" and

$$\overline{\tau}_\ell(f,g,h) \equiv \overline{(fgh)}_\ell - \overline{f}_\ell\overline{\tau}_\ell(g,h) - \overline{g}_\ell\overline{\tau}_\ell(f,h) - \overline{h}_\ell\overline{\tau}_\ell(f,g) - \overline{f}_\ell\overline{g}_\ell\overline{h}_\ell \tag{24}$$

is the 3^{rd}-order *generalized central moment* of fields $f(\mathbf{x}), g(\mathbf{x}), h(\mathbf{x})$ (see [25]). There are rigorous versions of relations (19)-(23). See [19] and Appendix A below for details.

Using relations (19) and (21), we can express baropycnal work as

$$\Lambda_\ell = \mathcal{O}\left[\frac{\delta P(\ell)}{\ell}\frac{\delta\rho(\ell)}{\overline{\rho}}\delta u(\ell)\right]. \tag{25}$$

In order to express Π_ℓ in terms of increments, we write down the following identities which are straightforward to verify:

$$\begin{aligned}
\tilde{\mathbf{u}} &= \overline{\mathbf{u}} + \overline{\tau}(\rho,\mathbf{u})/\overline{\rho} && (26)\\
\partial_j\tilde{u}_i &= \partial_j\overline{u}_i + \overline{\rho}^{-1}\partial_j\overline{\tau}(\rho,u_i) - \overline{\rho}^{-2}\overline{\tau}(\rho,u_i)\partial_j\overline{\rho} && (27)\\
\tilde{\tau}(u_i,u_j) &= \overline{\tau}(u_i,u_j) + \overline{\rho}^{-1}\overline{\tau}(\rho,u_i,u_j) - \overline{\rho}^{-2}\overline{\tau}(\rho,u_i)\overline{\tau}(\rho,u_j) && (28)
\end{aligned}$$

It then follows that we can express deformation work (12) as

$$\begin{aligned}
\Pi_\ell(\mathbf{x}) = &- \overline{\rho}\left[\partial_j\overline{u}_i + \overline{\rho}^{-1}\partial_j\overline{\tau}(\rho,u_i) - \overline{\rho}^{-2}\overline{\tau}(\rho,u_i)\partial_j\overline{\rho}\right]\\
&\times \left[\overline{\tau}(u_i,u_j) + \overline{\rho}^{-1}\overline{\tau}(\rho,u_i,u_j) - \overline{\rho}^{-2}\overline{\tau}(\rho,u_i)\overline{\tau}(\rho,u_j)\right]. && (29)
\end{aligned}$$

From (29) and relations (19),(21)-(23), we can express deformation work as

$$\begin{aligned}
\Pi_\ell = \mathcal{O}\Bigg[\ &\overline{\rho}\quad\left[\frac{\delta u(\ell)}{\ell} + \frac{\delta\rho(\ell)}{\overline{\rho}}\frac{\delta u(\ell)}{\ell} + \frac{\delta\rho^2(\ell)}{\overline{\rho}^2}\frac{\delta u(\ell)}{\ell}\right]\\
&\times\quad\left[\delta u^2(\ell) + \frac{\delta\rho(\ell)}{\overline{\rho}}\delta u^2(\ell) + \frac{\delta\rho^2(\ell)}{\overline{\rho}^2}\delta u^2(\ell)\right]\Bigg]. && (30)
\end{aligned}$$

Expressions (25) and (30) are not heuristic estimates, but are based on rigorous versions of (19)-(23), whose details can be found in [19] and in Appendix A.

[1] Relation (23) is based on an unpublished exact expression due to G. L. Eyink (see [20]). We repeat the details in Appendix A.

4 Locality of increments

Since Π_ℓ and Λ_ℓ can be expressed in terms of velocity, pressure, and density increments, it thus becomes sufficient to show that these increments themselves are scale-local. To establish this, we need the second requirement crucial for locality —that scaling properties of structure functions of velocity, pressure, and density increments are constrained by:

$$\begin{aligned} \|\delta\mathbf{u}(\mathbf{r})\|_p &\sim u_{rms}A_p(r/L)^{\sigma_p^u}, & 0<\sigma_p^u<1, \quad (31)\\ \|\delta P(\mathbf{r})\|_p &\sim P_{rms}B_p(r/L)^{\sigma_p^P}, & \sigma_p^P<1, \quad (32)\\ \|\delta\rho(\mathbf{r})\|_p &\leq \rho_{rms}C_p(r/L)^{\sigma_p^\rho}, & 0<\sigma_p^\rho \quad (33)\end{aligned}$$

for some dimensionless constants A_p, B_p, and C_p. The root-mean-square of a field $f(\mathbf{x})$ is denoted by $f_{rms}\equiv\langle f^2\rangle^{1/2}$, where $\langle\dots\rangle$ is a space average, $\frac{1}{V}\int d\mathbf{x}(\dots)$. Here, an L_p-norm $\|\cdot\|_p=\langle|\cdot|^p\rangle^{1/p}$ is just the traditional structure function $S_p=\langle|\cdot|^p\rangle$ raised to the $1/p$-th power. The scaling conditions (31)-(33) are well-established empirically in incompressible turbulent flows over intermediate scales $L\gg r\gg\ell_\mu$. They also have strong empirical support from many independent astronomical and numerical studies of compressible turbulent flows, which we discuss in detail in § 7.1. Note that condition (33) on the scaling of density increments is only an upper bound. It only stipulates that the intensity of density fluctuations decays at smaller scales, which is a very mild requirement and is readily satisfied in incompressible or nearly-incompressible flows.

Scaling conditions (31)-(33) reflect a structure of fields at intermediate scales. The constraints $\sigma_p^u<1$ and $\sigma_p^P<1$ indicate that velocity and pressure fields should be "rough enough" in a mean sense. They are not satisfied, for example, in laminar flows. These conditions are used to prove that contributions from the very large scales Δ to the flux across $\ell\ll\Delta$ are negligible or, in other words, that the flux is *infrared local.*

On the other hand, constraints $\sigma_p^u>0$ and $\sigma_p^\rho>0$ indicate that the velocity and density fields are "smooth enough" in a mean sense. They are not satisfied, for example, if the fields are dominated by small-scale fluctuations with a non-decaying spectrum. These conditions are used to prove that contributions from the very small scales δ to the flux across $\ell\gg\delta$ are negligible or, in other words, that the flux is *ultraviolet local.*

Notice that, unlike for the velocity and pressure fields, we do not stipulate that the density field be "rough enough." Contributions to the flux across scale ℓ from the largest density scales $L\gg\ell$ need not be negligible, yet the flux can still be scale-local.The underlying physical reason is simple; an energy flux across scale ℓ at a point $\mathbf{x}$ will depend on the mass in a ball of radius ℓ around $\mathbf{x}$. Mass is proportional to average density $\overline{\rho}_\ell(\mathbf{x})$ in the ball, which is dominated by large scales: $\overline{\rho}_\ell(\mathbf{x})=\mathcal{O}[\overline{\rho}_L(\mathbf{x})]=\mathcal{O}[\rho_{rms}]$.

Indeed, for incompressible turbulence, the only density scale present is a $\mathbf{k}=0$ Fourier mode, the largest possible, and the scale-local SGS flux is directly proportional to this density mode.

Furthermore, we do not require that the pressure field be "smooth enough," even though we expect $\sigma_p^P > 0$. This is because pressure only appears as a large-scale pressure-gradient in (13). For any filter kernel $\hat{G}(\mathbf{k})$ which is compact in Fourier space, contributions from wavenumbers $Q \gg \ell^{-1}$ will be exactly zero and ultraviolet locality of $\boldsymbol{\nabla}\overline{P}_\ell$ is guaranteed without any scaling assumptions.

Under conditions (31)-(33), proving scale locality of increments becomes simple and follows directly from [19]. For example, the contribution to any increment $\delta f(\ell)$ from scales $\Delta \geq \ell$ is represented by $\delta\overline{f}_\Delta(\ell)$. Here, $f(\mathbf{x})$ can denote either velocity or pressure field. Since the low-pass filtered field $\overline{f}_\Delta(\mathbf{x})$ is smooth, its increment may be estimated by Taylor expansion, and (19), and (31) or (32), as

$$\|\delta\overline{f}_\Delta(\ell)\|_p \simeq \|\boldsymbol{\ell}\boldsymbol{\cdot}(\boldsymbol{\nabla}\overline{f}_\Delta)\|_p = O\left[\ell\,\frac{1}{\Delta}\left(\frac{\Delta}{L}\right)^{\sigma_p^f}\right] = O\left[\left(\frac{\ell}{L}\right)^{\sigma_p^f}\left(\frac{\ell}{\Delta}\right)^{1-\sigma_p^f}\right], \tag{34}$$

and this is negligible for $\Delta \gg \ell$ as long as $\sigma_p^f < 1$. The notation $O(\dots)$ denotes a big-O upper bound.

On the other hand, the contribution to any increment $\delta f(\ell)$ from scales $\delta \leq \ell$ is represented by $\delta f'_\delta(\ell)$. Here, $f(\mathbf{x})$ can denote either velocity or density field. Since $f'_\delta = \mathcal{O}[f(x+\delta)-f(x)]$ from (20), scaling conditions (31) and (33) imply that

$$\|\delta f'_\delta(\ell)\|_p = O\left[\left(\frac{\delta}{L}\right)^{\sigma_p^f}\right] = O\left[\left(\frac{\ell}{L}\right)^{\sigma_p^f}\left(\frac{\delta}{\ell}\right)^{\sigma_p^f}\right], \tag{35}$$

and this is negligible for $\delta \ll \ell$ as long as $\sigma_p^f > 0$. For more details and for the careful proofs of these statements, see [19].

5 Locality of flux

Based on results in § 3 and § 4, proving scale locality of the flux terms Π_ℓ and Λ_ℓ is straightforward. To illustrate, consider flux due to baropycnal work, Λ_ℓ in (13). It is a quartic quantity which depends on two density modes, one pressure mode, and one velocity mode. This dependence can be made more explicit by writing

$$\Lambda_\ell(\rho, P, \rho, \mathbf{u}) \equiv \frac{1}{\overline{\rho}_\ell}\boldsymbol{\nabla}\overline{P}_\ell\boldsymbol{\cdot}\overline{\tau}_\ell(\rho, \mathbf{u}),$$

where the first density argument $\Lambda(\rho, ., ., .)$ corresponds to the factor $1/\overline{\rho}$.

To prove ultraviolet locality of Λ_ℓ, we need to show that contributions from each of the four arguments $(\rho, P, \rho, \mathbf{u})$ at scales $\delta \ll \ell$ is negligible. It is obvious that $\overline{\rho}_\ell$ will have vanishing contribution (decaying faster than any power, or exactly zero for filter kernels $\hat{G}(\mathbf{k})$ compact in Fourier space) from scales $\delta \ll \ell$.

It follows that its reciprocal $1/\overline{\rho}_\ell$ also has vanishing[2] contribution from very small scales $\delta \ll \ell$. It is also obvious that $\boldsymbol{\nabla}\overline{P}_\ell$ will have vanishing contribution from scales $\delta \ll \ell$.

What remains to be shown is that the last two arguments $(.,.,\rho,\mathbf{u})$ have negligible ultraviolet contributions. If we replace them by ρ'_δ and $\mathbf{u}'_\delta$, respectively, we get from using Hölder's inequality that

$$\begin{aligned} \|\Lambda_\ell(\rho,P,\rho'_\delta,\mathbf{u}'_\delta)\|_p &\leq (\text{const.})\|\frac{1}{\overline{\rho}_\ell}\|_\infty \|\boldsymbol{\nabla}\overline{P}_\ell\|_{3p}\|\delta\rho'_\delta(\ell)\|_{3p}\|\delta u'_\delta(\ell)\|_{3p} \\ &= O\left[\left(\frac{\ell}{L}\right)^{\sigma^P_{3p}+\sigma^\rho_{3p}+\sigma^u_{3p}-1}\left(\frac{\delta}{\ell}\right)^{\sigma^\rho_{3p}+\sigma^u_{3p}}\right], \end{aligned} \tag{36}$$

which vanishes as $\delta/\ell \to 0$ for any L_p-norm if $\sigma^\rho_{3p}+\sigma^u_{3p} > 0$. We used relation (21) to get $\|\overline{\tau}_\ell(\rho'_\delta,\mathbf{u}'_\delta)\|_{3p/2} \leq (\text{const.})\|\delta\rho'_\delta(\ell)\|_{3p}\|\delta u'_\delta(\ell)\|_{3p}$ and result (35) to arrive at the upper bound. A slight complication in our result is the additional overall factor $(\ell/L)^\alpha$, where $\alpha \equiv \sigma^P_{3p}+\sigma^\rho_{3p}+\sigma^u_{3p}-1$. This can grow with decreasing ℓ if $\alpha < 0$, causing our upper bound (36) to deteriorate at small scales. Such a growth, however, can be easily offset by taking δ small enough: $\delta < \delta_*(\ell) = \ell\,(\ell/L)^{(1-\sigma^P_{3p}-\sigma^\rho_{3p}-\sigma^u_{3p})/(\sigma^\rho_{3p}+\sigma^u_{3p})}$. We finally conclude that flux term Λ_ℓ is ultraviolet local under scaling conditions (31),(33) for the velocity and density fields.

Next we prove infrared locality of $\Lambda_\ell(\rho,P,\rho,\mathbf{u})$ by replacing the pressure and velocity arguments with $\overline{P}_\Delta$ and $\overline{\mathbf{u}}_\Delta$, respectively, for $\Delta \gg \ell$. Using Hölder's inequality, we get

$$\begin{aligned} \|\Lambda_\ell(\rho,\overline{P}_\Delta,\rho,\overline{\mathbf{u}}_\Delta)\|_p &\leq (\text{const.})\|\frac{1}{\overline{\rho}_\ell}\|_\infty \|\delta\overline{P}_\Delta(\ell)/\ell\|_{3p}\|\delta\rho(\ell)\|_{3p}\|\delta\overline{u}_\Delta(\ell)\|_{3p} \\ &= O\left[\left(\frac{\ell}{L}\right)^{\sigma^P_{3p}+\sigma^\rho_{3p}+\sigma^u_{3p}-1}\left(\frac{\ell}{\Delta}\right)^{2-\sigma^P_{3p}-\sigma^u_{3p}}\right], \end{aligned} \tag{37}$$

which vanishes as $\ell/\Delta \to 0$ for any L_p-norm if $\sigma^P_{3p}+\sigma^u_{3p} < 2$. We used relations (19) and (21) to get the inequality and result (34) to arrive at the upper bound. As in the ultraviolet case, there is a factor $(\ell/L)^\alpha$ which can cause our upper bound (37) to increase at small ℓ if $\alpha < 0$. Again, such an increase can be easily compensated by taking Δ large enough: $\Delta > \Delta_*(\ell) = \ell\,(\ell/L)^{(\sigma^P_{3p}+\sigma^\rho_{3p}+\sigma^u_{3p}-1)/(2-\sigma^P_{3p}-\sigma^u_{3p})}$. We finally conclude that flux term Λ_ℓ is infrared local under scaling conditions (31),(32) for the velocity and pressure fields.

Similarly, we can derive rigorous upper bounds on non-local contributions for each of the 9 terms in (29) to prove scale locality of flux due to deformation work, Π_ℓ.

6 Locality of the cascade

In proving locality of the SGS flux, $\Pi_\ell+\Lambda_\ell$, we did not need to make any assumptions about an equation of state for the fluid. We also did not analyze the internal energy budget (3). It is not obvious to us how

[2] For any positive smooth filter kernel, the field $\overline{\rho}_\ell(\mathbf{x})$ is real analytic and non-zero for all $\mathbf{x}$. A well-known result from real analysis (see for e.g. [33]) states that the reciprocal $1/\overline{\rho}_\ell$ is also real analytic and, therefore, its Fourier mode amplitudes decay faster than any power n of wavenumber k^{-n} as $k \to \infty$.

to best define a notion of scale for internal energy consistent with our scale-decomposition of the flow field in [1]. Despite this shortcoming, we were still able to derive important results concerning locality.

Circumventing the aforementioned shortcoming was possible due to two facts. First, we proved rigorously in [1] that viscous dissipation is negligible at large-scales, which implies that large-scale kinetic energy does not couple to internal energy via viscous dynamics. The only coupling that exists is via large-scale pressure dilatation. The second fact which aided us in proving locality of the SGS flux is that large-scale pressure dilatation, $-\overline{P}_\ell \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell$, does not involve scales $< \ell$ and, therefore, it cannot transfer kinetic energy *directly across* scales. Hence, pressure dilatation does not contribute to the SGS kinetic energy flux.

In principle, one could conjure a scenario in which mean large-scale kinetic energy is converted to internal energy at scale $\sim \ell_1$ via $PD(\ell) \equiv -\langle \overline{P}_\ell \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell \rangle$ and is subsequently re-converted back, *indirectly*, into mean kinetic energy at a much smaller scale $\sim \ell_2 \ll \ell_1$. In other words, as we continuously probe smaller scales ℓ, the function $PD(\ell)$ is positive at $\sim \ell_1$, then decreases and becomes zero at $\sim \ell_2$. The problem arises if such a process repeats itself, whereby $PD(\ell)$ oscillates indefinitely with a non-decaying amplitude, as $\ell \to 0$.

As unlikely as such a scenario may appear, we do not know of a rigorous argument to disprove it under the weak assumptions (31)-(33) we have already made. It, therefore, precludes us from claiming at this point in the paper that our proof of a scale-local SGS flux implies rigorously a scale-local cascade process in compressible turbulence. It is possible, however, to infer rigorously that the cascade is scale-local if we make one additional assumption which is, albeit reasonable, not as weak as scaling conditions (31)-(33).

6.1 Pressure dilatation co-spectrum

The assumption we need to prove a scale-local cascade concerns the pressure dilatation co-spectrum, defined as

$$E^{PD}(k) \equiv \sum_{k-0.5<|\mathbf{k}|<k+0.5} -\hat{P}(\mathbf{k})\widehat{\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u}}(-\mathbf{k}), \tag{38}$$

which we require to decay at a fast enough rate,

$$|E^{PD}(k)| \leq C\, u_{rms}\, P_{rms}\, (kL)^{-\beta}, \qquad \beta > 1. \tag{39}$$

Here, C is a dimensionless constant and L is an integral scale.

In the limit of infinite Reynolds number, assumption (39) implies that *mean* pressure dilatation, $PD(\ell)$, asymptotes to a finite constant, $\theta \equiv -\langle P\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u}\rangle$, as $\ell \to 0$. In other words, mean pressure

dilatation, $PD(\ell)$, acting at scales $> \ell$ converges and becomes independent of ℓ at small enough scales:

$$\lim_{\ell\to 0} PD(\ell) = \lim_{K\to\infty} \sum_{k<K} E^{PD}(k) = \theta, \tag{40}$$

for wavenumber $K \approx \ell^{-1}$. In § 7.2, we shall give a physical argument on why we expect (40) to hold.

We remark that condition (39) is sufficient but not necessary for the convergence of $PD(\ell)$ in the limit of $\ell \to 0$. It is possible for $E^{PD}(k)$, which is not sign-definite, to oscillate around 0 as a function of k, such that the series $\sum_{k<K} E^{PD}(k)$ converges with $K \to \infty$ at a rate faster than what is implied by assumption (39).

6.2 Conservative kinetic energy cascade

Saturation of *mean* pressure dilatation, as expressed in (40), reveals that its role is to exchange *large-scale* mean kinetic and internal energy over a transitional "conversion" scale-range. At smaller scales beyond the conversion range, mean kinetic and internal energy budgets statistically decouple. In other words, taking $\ell_\mu \to 0$ first, then $\ell \to 0$, the steady-state mean kinetic energy budget becomes,

$$\langle \Pi_\ell + \Lambda_\ell \rangle = \langle \epsilon^{inj} \rangle - \theta. \tag{41}$$

We stress that such a decoupling is statistical and does not imply that small scales evolve according to incompressible dynamics. However, while small-scale compression and rarefaction can still take place pointwise, they yield a vanishing contribution to the *space-average*.

We denote the largest scale at which such statistical decoupling occurs by ℓ_c. It may be defined, for instance, as the scale at which $PD(\ell_c) = 0.95\,\theta$. Alternatively, it may be defined as

$$\ell_c \equiv \frac{\sum_{\mathbf{k}} k^{-1} E^{PD}(\mathbf{k})}{\sum_{\mathbf{k}} E^{PD}(\mathbf{k})}. \tag{42}$$

Over the ensuing scale-range, $\ell_c > \ell \gg \ell_\mu$, net pressure dilatation does not play a role, and if, furthermore, $\langle \epsilon^{inj} \rangle$ in (41) is localized to the largest scales as shown in [1], then $\langle \Pi_\ell + \Lambda_\ell \rangle$ will be a constant, independent of scale ℓ.

A constant SGS flux implies that mean kinetic energy cascades conservatively to smaller scales, despite not being an invariant of the governing dynamics. This is one of the major conclusions of our paper. In particular, the scenario discussed in the two paragraphs preceding § 6.1 cannot ocure over $\ell_c > \ell \gg \ell_\mu$, and kinetic energy can only reach dissipation scales via the SGS flux, $\Pi_\ell + \Lambda_\ell$, through a scale-local cascade process. We are therefore justified in calling scale-range $\ell_c > \ell \gg \ell_\mu$ the inertial range of compressible turbulence.

7 Discussion

In proving locality of the SGS flux, $\Pi_\ell + \Lambda_\ell$, we did not invoke assumptions of homogeneity or isotropy. We only assumed the weak scaling conditions (31)-(33) on structure functions of velocity, pressure, and density. The results also apply to individual realizations of the flow, without the need for ensemble averaging.

7.1 Validity of assumptions (31)-(33)

We have proved through an exact analysis of the fluid equations that scaling assumptions (31)-(33) are sufficient to guarantee scale locality of SGS flux. The ultimate source of these scaling properties is empirical evidence from experiments, astronomical observations, and numerical simulations [3].

For incompressible turbulence, which may be viewed as a limiting case of our analysis, the scaling of velocity and pressure structure functions (31),(32) has been well-established by a variety of independent studies such as those by [4, 8, 12, 49, 27, 26, 51]. Assumption (33) on density structure functions is trivially satisfied for a uniform density field.

As for compressible turbulence, the available data is also in compelling support of our assumptions. Astronomical observations by [5, 6] of radio wave scintillation in the interstellar medium, characterized by highly supersonic turbulent flows, possibly up to Mach 20 (see for example [42]), show that 2nd-order density structure function scales with $\sigma_2^\rho \doteq 0.3$ over 5 decades in scale. [50, 7] used velocity integrated spectral line maps of several molecular clouds and found evident power-law scaling for the density with exponent $0.3 \leq \sigma_2^\rho \leq 0.4$. Much effort has also been expended to measure scaling of 2nd-order velocity structure functions. Using spectroscopic surveys of molecular clouds, which give line-of-sight velocities from emission lines, several independent studies by []Falgaroneetal92,MieschBally94,BruntHeyer02; [41] found scaling exponents $\sigma_2^u \doteq 0.4$. More recently, [28] measured scaling of structure functions up to 6th-order and found $\sigma_1^u \doteq 0.54$, $\sigma_2^u \doteq 0.51$, $\sigma_3^u \doteq 0.49$, $\sigma_4^u \doteq 0.47$, $\sigma_5^u \doteq 0.46$, $\sigma_6^u \doteq 0.45$. Analysis of solar wind data has also yielded $0 < \sigma_p^u < 1$, for $1 \leq p \leq 6$ (see for example []Podestaetal07 and [46]).

Alongside observational evidence, [34, 48, 47, 23, 45] carried out several independent numerical studies of forced compressible turbulence and calculated scaling of structure functions. Their simulations employed an isothermal equation of state and reached a range of high turbulent Mach numbers (based on u_{rms} of velocity fluctuations), $M_t = 2.5 - 10$. They report power-law scaling exponents well within our required constraints, $0 < \sigma_p^\rho$ and $0 < \sigma_p^u < 1$ for $1 \leq p \leq 6$. One caveat of such simulations is that they do not resolve viscous dynamics explicitly but rely on numerical schemes to deal with shocks and dissipation. Based on such considerations, they may not be deemed direct numerical simulations

[3] Our scaling conditions (31)-(33) do not distinguish between compressive $\mathbf{u}^c$ and solenoidal $\mathbf{u}^s$ components of the velocity field. Whereas the contribution to structure functions in the incompressible limit will be predominantly from $\mathbf{u}^s$, contributions from $\mathbf{u}^c$ may be significant in general, as in the case of Burger's turbulence.

but rather implicit LES with an uncontrolled subgrid model and, therefore, are not as reliable. DNS of compressible turbulence cannot achieve simultaneously high Reynolds and Mach numbers due to resolution limitations. The largest DNS to date we are aware of is that by [43]. It was on a 1024^3 grid, had a Taylor Reynolds number $Re_\lambda = 300$ and a turbulent Mach number $M_t = 0.3$ —still in the subsonic regime.

We can also gain useful insights into the weakness of conditions (31)-(33) through exact mathematical considerations. From the definition of a structure function, $S_p^f(\ell) = \langle|\delta f(\ell)|^p\rangle$, for any field $f(\mathbf{x})$ and its scaling exponent, $\zeta_p^f = \liminf_{\ell\to 0}[\ln S_p^f(\ell)/\ln(\ell/L)]$ (see [19, 17, 18]), it is known that ζ_p^f is a concave function of $p \in [0,\infty)$ (see for example [24, 20] for details). It follows that our scaling exponents $\sigma_p^f = \zeta_p^f/p$ are non-increasing functions of p (see [20]). For example, if we have $\sigma_1 < 1$, then we are guaranteed $\sigma_p < 1$ for any $p \geq 1$. Similarly, if $\sigma_q > 0$ for some $q > 1$, then $\sigma_p > 0$ for any $p \leq q$. Another known result states that if the pth-order moment of $f(\mathbf{x})$ exists, $\langle|f|^p\rangle < \infty$ for $p \geq 1$, then the pth-order scaling exponent is non-negative, $\sigma_p^f \geq 0$ (see [24, 20]).

To further put our scaling assumptions into perspective, a 2nd-order structure function $S_2^u(\ell) \sim \ell^{2\sigma_2^u}$ is related to the spectrum $E^u(k) \sim k^{-n}$ with $n = 2\sigma_2^u + 1$. Therefore, condition $\sigma_2^u > 0$ is equivalent to a constraint on the spectral exponent $n > 1$. This condition on n is the same as that required for a stationary velocity field in a bounded domain to have finite variance, $\langle|\mathbf{u}|^2\rangle < \infty$, and a power-law spectrum k^{-n} for $k \in [k_0,\infty)$ in the limit $Re \to \infty$ (see [24]).

7.2 Validity of assumption (39)

The existence of a scale-local cascade over an inertial range is the main conclusion of this paper. However, to reach our result, we made the important assumption (39) which deserves more careful examination. Needless to say, the scaling of pressure dilatation co-spectrum is easily measurable from numerical simulations. The only reported measurement of this quantity we are aware of is by []Leeetal06 shown in their figure 6(b). The authors had the same purpose in mind; to check the scales at which pressure dilatation exchanges kinetic and internal energy. From their plot, they concluded that such an exchange takes place only at the largest scales. While their conclusion is in support of our postulate, their plot is on a log-linear scale which precludes the inference of such a conclusion. It is possible for the co-spectrum to scale with $\beta \leq 1$ in (39) while seeming to have most of its contribution from the largest scales. The point we want to emphasize here is that it is $PD(\ell)$, the integral of the co-spectrum, which needs converge and have most of its contribution from the largest scales. We note that our condition (39) does not require a power-law scaling —only that $E^{PD}(k)$ decays at a rate faster than $\sim k^{-1}$.

It is not at all trivial why one should expect $PD(\ell) = -\langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle$ to converge at small scales. How can this be reconciled with the expectation that compression, as quantified by $\boldsymbol{\nabla}\cdot\mathbf{u}$, would get more

intense at smaller scales? Indeed, it has been observed numerically that $(\nabla\cdot\mathbf{u})_{rms}$ is an increasing function of Reynolds number (see for example []Leeetal91). The key point here is that our assumption (39) concerns *spatially averaged* pressure dilatation. It is true that $\nabla\cdot\mathbf{u}(\mathbf{x})$, being a gradient, derives most of its contribution from the smallest scales in the flow. However, since $P\nabla\cdot\mathbf{u}$ is not sign-definite, major cancellations can occur when space-averaging. The situation is very similar to helicity co-spectrum in incompressible turbulence, $H(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \hat{\mathbf{u}}(\mathbf{k})\cdot\widehat{\nabla\times\mathbf{u}}(-\mathbf{k})$. Here, the pointwise vorticity, $\boldsymbol{\omega}(\mathbf{x}) = \nabla\times\mathbf{u}$, can also become unbounded in the limit of infinite Reynolds number. However, numerical evidence shows that $\langle\mathbf{u}\cdot\boldsymbol{\omega}\rangle$ remains finite with Reynolds number and the co-spectrum $H(k)$ decays at a rate $\sim k^{-n}$, with $n \approx 5/3 > 1$ (see for example [11] and []Kurienetal04).

We can offer a physical argument on why $PD(\ell)$ is expected to converge for $\ell \to 0$ as a result of cancellations from space-averaging. The origin of such cancellations can be heuristically explained using decorrelation effects very similar to those studied in [21] and [2]. While the pressure in $\langle P\nabla\cdot\mathbf{u}\rangle$ derives most of its contribution from the largest scales, $\nabla\cdot\mathbf{u}$ is dominated by the smallest scales in the flow. Therefore, pressure varies slowly in space, primarily at scales $\sim L$, while $\nabla\cdot\mathbf{u}$ varies much more rapidly, primarily at scales $\ell_\mu \ll L$, leading to a decorrelation between the two factors. More precisely, the pressure $\overline{P}_\ell$ in $PD(\ell)$ may be approximated by $\overline{P}_\ell = \mathcal{O}[P_{rms}] = \mathcal{O}[\overline{P}_L]$ such that

$$\langle\overline{P}_\ell\nabla\cdot\overline{\mathbf{u}}_\ell\rangle \approx \left\langle\overline{P}_L\nabla\cdot\left(\overline{(\overline{\mathbf{u}}_\ell)}_L + (\overline{\mathbf{u}}_\ell)'_L\right)\right\rangle \quad\approx\quad \langle\overline{P}_L\nabla\cdot\overline{\mathbf{u}}_L\rangle + \langle\overline{P}_L\rangle\langle\nabla\cdot(\overline{\mathbf{u}}_\ell)'_L\rangle.$$

The first term in the last expression follows from $\overline{(\overline{\mathbf{u}}_\ell)}_L \approx \overline{\mathbf{u}}_L$, while the second term is due to an approximate statistical independence between $\overline{P}_L$ and $\nabla\cdot(\overline{\mathbf{u}}_\ell)'_L \sim \delta u(\ell)/\ell$ which varies primarily at much smaller scales $\sim \ell \ll L$. If there is no transport beyond the domain boundaries or if the flow is either statistically homogeneous or isotropic, we get $\langle\nabla\cdot(\overline{\mathbf{u}}_\ell)'_L\rangle = 0$. The heuristic argument finally yields that pressure dilatation,

$$PD(\ell) = -\langle\overline{P}_\ell\nabla\cdot\overline{\mathbf{u}}_\ell\rangle \approx -\langle\overline{P}_L\nabla\cdot\overline{\mathbf{u}}_L\rangle, \tag{43}$$

becomes independent of ℓ, for $\ell \ll L$. Expression (43) corroborates our claim that the primary role of mean pressure dilatation is conversion of *large-scale* kinetic energy into internal energy and does not participate in the cascade dynamics beyond a transitional "conversion" scale-range.

7.3 A related study

An insightful and clever numerical study by [36], which we mentioned in § 7.2, came to our attention at an advanced stage of writing this paper. The authors carried out DNS of compressible isotropic turbulence at low Mach number decaying under the influence of a randomly distributed temperature field. Some of the main conclusions are elegantly summarized by a schematic in their figure 8. They assert that mean pressure dilatation only acts at the largest scales beyond which mean kinetic energy

cascades conservatively down to the viscous scales where it is dissipated into heat. This conclusion is identical to the picture we arrived at in § 6.2 and § 7.2. However, their study does not address the issue of scale locality which forms a main theme of our paper.

On the other hand, their work goes beyond these statements and maintains that mean pressure dilatation couples internal energy to irrotational (and not solenoidal) modes of the velocity field. They also observe that mean pressure dilatation is not sign-definite in time but tends to transfer energy from internal to kinetic energy after time-averaging. Furthermore, they contend that the coupling of solenoidal and irrotational modes of the velocity field is weak and that each cascades separately to the viscous scales. They also investigate the alignments between vorticity and gradients of pressure, density, and temperature. All of these are essential issues which we do not tackle in our paper.

The paper of [36] is a very valuable numerical investigation of the fundamental physics of compressible turbulence. Yet, inspection of dissipation spectra in their figures 6(c,d) seem to indicate that the turbulent flows they studied were not fully developed. We believe that similar studies at higher Reynolds and Mach numbers, and under different controlled conditions are still needed to establish the findings of [36] as empirical facts.

7.4 Intermittency, locality, and universality

As we mentioned above, scale locality is necessary to justify the notion of universality of intertial-range statistics. Kolmogorov's original 1941 theory of incompressible turbulence assumed statistical self-similarity of inertial-range scales. Today, there is a general consensus based on substantial empirical evidence that fluctuations in turbulent flows are not statistically self-similar but are subject to intermittency corrections. For p-th order structure functions, this is expressed as

$$S_p(\ell) \equiv \langle|\delta u(\ell)|^p\rangle \sim u_{rms}^p \left(\frac{\ell}{L}\right)^{\zeta_p} = (\langle\epsilon\rangle\ell)^{p/3}\left(\frac{\ell}{L}\right)^{\delta\zeta_p}, \tag{44}$$

where average energy flux $\langle\epsilon\rangle$ (or dissipation) is empirically related to u_{rms} and L through the zeroth-law of incompressible turbulence, $\langle\epsilon\rangle = u_{rms}^3/L$, and we have used $\zeta_p = p/3 + \delta\zeta_p$. Relation (44) shows that for non-zero "anomalous exponents" $\delta\zeta_p$, statistical averages at inertial-range scales $\ell \ll L$ are a function of integral length, L. Colloquially, this means that inertial-range scales "remember" the number of "cascade steps", $\log_2(L/\ell)$, for energy in going from scale L to scale ℓ.

However, intermittency does not contradict scale locality or the existence of universal scaling. It is known, for example, that the GOY shell model, in which scale interactions are local by construction, exhibits intermittency (see for example [9]). Despite remembering the number of cascade steps, there is no *direct* communication between inertial and integral length-scales due to scale locality. It is precisely because individual cascade step are scale-local and depend only upon inertial-range dynamics, that it

is possible to argue for universality of the scaling exponents ζ_p, regardless of (and consistently with) intermittency corrections. For a more detailed discussion of such issues, see [20].

7.5 Shocks and locality

An idea especially common in the astrophysical literature claims that in compressible turbulence a "finite portion" of energy at a given scale must be dissipated directly into heat via shocks rather than cascading in a local fashion (see for example [38]). Our analysis shows that large-scale kinetic energy can only reach dissipation scales through SGS flux, $\Pi_\ell + \Lambda_\ell$, which we have have proved to be scale-local provided the weak scaling conditions (31)-(33) are satisfied. Therefore, in order for large-scale kinetic energy to dissipate into heat non-locally, it is necessary to violate (31),(33) such that $\sigma_p^u \leq 0$ or $\sigma_p^\rho \leq 0$ for $p \leq 7$ to break down ultraviolet locality. Having $\sigma_p^{u,\rho} = 0$ implies that the mean intensity of velocity or density fluctuations does not decay at smaller scales. All empirical evidence discussed in subsection 7.1 seems to rule out such a possibility.

The situation in compressible turbulence is similar to that of incompressible MHD turbulence where discontinuities in the magnetic field, i.e. current sheets, are pervasive. However, [3] proved rigorously, under scaling conditions analogous to (31)-(33), that the cascade is local in scale and, furthermore, provided numerical support from high-resolution DNS.

Another elucidating example is that of Burger's turbulence, in which viscous dissipation takes place only in shocks (which have zero volume when the Reynolds number is infinite) and vanishes everywhere else. However, scaling exponents of velocity increments are known to be $\sigma_p^u = 1/p$ for $p \geq 1$, which satisfy the condition $0 < \sigma_p^u < 1$ for any $1 \leq p < \infty$, and the proof of locality applies to Burger's energy cascade as well. In fact, the same conclusion was pointed out by [32] where he stated that the cascade in turbulence can be scale-local despite the presence of coherent discontinuous structures. He specifically discussed Burger's turbulence and showed that the energy is transfered to smaller scales in a local cascade process. As we mentioned in the introduction § 1, Kraichnan at the time had realized through his closure theory that scale locality depends on the exponent of the spectrum power-law and not on coherence properties. Furthermore, scale locality is perfectly consistent with the possibility that all dissipation takes place only in shocks and singular structures, over a set of zero volume in the limit of $\mu \to 0$. It is well-known for this to be the case in Burger's flow. The point here being that scale locality is an inertial-range property of SGS flux which transfers energy across scales and not of viscous dissipation which converts energy into heat.

8 Summary

We proved that in compressible turbulence, the SGS flux responsible for direct transfer of kinetic energy across scales is dominated by scale-local interactions. This was achieved through rigorous upper bounds on the non-local contributions, and under weak assumptions (31)-(33). No assumptions of homogeneity or isotropy were invoked and the results hold for any equation of state of the fluid.

We also showed that, in the limit of high Reynolds number, scale locality of the SGS flux implies a scale-local cascade of kinetic energy if condition (39) on pressure dilatation co-spectrum is satisfied. In particular, condition (39) implies that beyond a transitional "conversion" scale-range, there exists an inertial scale-range over which mean kinetic and internal energy budgets statistically decouple and the mean SGS flux, $\langle \Pi_\ell + \Lambda_\ell \rangle$, becomes a constant, independent of scale ℓ. Our result in § 6 demonstrates that kinetic energy cascades conservatively despite not being an invariant.

We remark that the extent of the conversion range could be an increasing function of Mach number and/or the ratio of compressive-to-solenoidal kinetic energy. If so, it would have an immediate bearing on the interpretation of results such as those in figure 2 of [40], where the Mach number is varied while maintaining a fixed Reynolds number. In such studies, an increasing Mach number may lead to the conversion range eroding away the finite inertial range present in simulations. Measurements of power-law exponents over the conversion range may not reflect an asymptotic scaling which would otherwise appear at sufficiently high Reynolds numbers. The problem is that of an ordering of limits; the physically interesting order being one in which we take $Re \to \infty$ first, followed by $M_t \to \infty$.

In summary, we conclude that there exists an inertial range in high Reynolds number compressible turbulence over which kinetic energy reaches dissipation scales through a conservative and scale-local cascade process. This precludes the possibility for transfer of kinetic energy from the large-scales directly to dissipation scales, such as into shocks, at arbitrarily high Reynolds numbers as is commonly believed. Our work makes several assumptions and predictions which can be tested numerically. Our locality results concerning the SGS flux can be verified in a manner very similar to what was done in [2] and [3]. We also invite numerical tests of assumption (39) on the scaling of pressure dilatation co-spectrum. Verifying (39) or (40) under a variety of controlled conditions would substantiate the idea of statistical decoupling between mean kinetic and internal energy budgets. This would have potentially significant implications on devising reduced models of compressible turbulence as well as providing physical insight into this rich problem.

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A Details of relations (19)-(23)

We repeat from [19] and [20] the details of expressing gradients and central moments in terms of increments.

Relation (19) is a result of integration by parts and rewriting a large-scale gradient as:

$$\begin{aligned} \boldsymbol{\nabla}\overline{f}_\ell(\mathbf{x}) &= -\frac{1}{\ell}\int d\mathbf{r}(\boldsymbol{\nabla}G)_\ell(\mathbf{r})f(\mathbf{x}+\mathbf{r}) \\ &= -\frac{1}{\ell}\int d\mathbf{r}(\boldsymbol{\nabla}G)_\ell(\mathbf{r})[f(\mathbf{x}+\mathbf{r})-f(\mathbf{x})], \end{aligned} \tag{45}$$

where $(\boldsymbol{\nabla}G)_\ell(\mathbf{r}) = \ell^{-d}(\boldsymbol{\nabla}G)(\mathbf{r}/\ell)$ in d-dimensions and we have used condition $\int d\mathbf{r}\boldsymbol{\nabla}G(\mathbf{r}) = \mathbf{0}$ to arrive at the last equality.

High-pass filtered fields in relation (20) can be expressed in terms of increments as:

$$\begin{aligned} f'_\ell(\mathbf{x}) &= -\int d\mathbf{r}\, G_\ell(\mathbf{r})[f(\mathbf{x}+\mathbf{r})-f(\mathbf{x})] \\ &= -\langle \delta f(\mathbf{x};\mathbf{r})\rangle_\ell, \end{aligned} \tag{46}$$

where $\langle \delta f(\mathbf{x};\mathbf{r})\rangle_\ell \equiv \int d\mathbf{r} G_\ell(\mathbf{r})\delta f(\mathbf{x};\mathbf{r})$ is an average over separations $\mathbf{r}$ in a ball of radius of order ℓ centered at $\mathbf{x}$.

The 2^{nd}-order central moment relation (21) is due to []Constantinetal94 and is straightforward to verify:

$$\overline{\tau}_\ell\left(f\left(\mathbf{x}\right), g\left(\mathbf{x}\right)\right) = \langle \delta f(\mathbf{x};\mathbf{r})\delta g(\mathbf{x};\mathbf{r})\rangle_\ell - \langle \delta f(\mathbf{x};\mathbf{r})\rangle_\ell \langle \delta g(\mathbf{x};\mathbf{r})\rangle_\ell. \tag{47}$$

Relation (22) follows from (47) through an integration by parts:

$$\begin{aligned} \boldsymbol{\nabla}\overline{\tau}_\ell\left(f\left(\mathbf{x}\right), g\left(\mathbf{x}\right)\right) = -\frac{1}{\ell}\Big\{ &\int d\mathbf{r}(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\delta f(\mathbf{x};\mathbf{r})\delta g(\mathbf{x};\mathbf{r}) \\ &-\int d\mathbf{r}(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\delta f(\mathbf{x};\mathbf{r})\int d\mathbf{r}' G_\ell(\mathbf{r}')\delta g(\mathbf{x};\mathbf{r}') \\ &-\int d\mathbf{r} G_\ell(\mathbf{r})\delta f(\mathbf{x};\mathbf{r})\int d\mathbf{r}'(\boldsymbol{\nabla}G)_\ell(\mathbf{r}')\delta g(\mathbf{x};\mathbf{r}')\Big\}. \end{aligned} \tag{48}$$

The 3^{rd}-order central moment in relation (23) is due to [20] and is straightforward to verify:

$$\begin{aligned}\overline{\tau}_\ell(f,g,h)(\mathbf{x}) &= \langle\delta f(\mathbf{x};\mathbf{r})\delta g(\mathbf{x};\mathbf{r})\delta h(\mathbf{x};\mathbf{r})\rangle_\ell \\ &- \langle\delta f(\mathbf{x};\mathbf{r})\rangle_\ell\langle\delta g(\mathbf{x};\mathbf{r})\delta h(\mathbf{x};\mathbf{r})\rangle_\ell \\ &- \langle\delta g(\mathbf{x};\mathbf{r})\rangle_\ell\langle\delta f(\mathbf{x};\mathbf{r})\delta h(\mathbf{x};\mathbf{r})\rangle_\ell \\ &- \langle\delta h(\mathbf{x};\mathbf{r})\rangle_\ell\langle\delta f(\mathbf{x};\mathbf{r})\delta g(\mathbf{x};\mathbf{r})\rangle_\ell \\ &+ 2\langle\delta f(\mathbf{x};\mathbf{r})\rangle_\ell\langle\delta g(\mathbf{x};\mathbf{r})\rangle_\ell\langle\delta h(\mathbf{x};\mathbf{r})\rangle_\ell. \end{aligned} \tag{49}$$

B Locality proofs

The proofs for locality follow closely those of [19]. We will repeat briefly the ones which were explicitly discussed in the latter work and also give some details on locality of new terms not present in the [19] treatment.

The filter kernel $G(\mathbf{r})$ used in the proofs is smooth and decays faster than any power r^{-p} as $r\to\infty$. It is also possible (but not required) to have both it and its Fourier transform $\hat{G}(\mathbf{k})$ be positive and, furthermore, to have the latter compactly supported inside a ball of radius 1 about the origin in Fourier space.

B.1 Ultraviolet locality

Ultraviolet locality means that contributions to the energy flux across ℓ from scales $\delta\ll\ell$ decay at least as fast as δ^α, for some $\alpha>0$. We will now show that each of the factors in SGS flux terms (13) and (29) is ultraviolet local.

Non-local ultraviolet contributions to a large-scale gradient of field $f(\mathbf{x})$ can be shown to be bounded using (45)-(46) as was proved by [19]:

$$\|\boldsymbol{\nabla}\overline{f'_\delta}\|_p \le \frac{2}{\ell}\int d\mathbf{r}\,|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})|\;\|f'_\delta(\mathbf{x})\|_p = O\left(\frac{\delta^{\sigma^f_p}}{\ell}\right). \tag{50}$$

Notation $O(\dots)$ denotes a big-O upper bound. In fact, $\|\boldsymbol{\nabla}\overline{\mathbf{u}'_\delta}\|_p=0$ for a filter $\hat{G}(\mathbf{k})$ compactly supported in Fourier space.

Non-local ultraviolet contributions to a second-order central moment of fields $f(\mathbf{x})$ and $g(\mathbf{x})$ can be bounded using (46),(47) as was proved by [19]. For $1/p=1/r+1/s$, we have

$$\begin{aligned}\|\overline{\tau}_\ell(f'_\delta,g'_\delta)\|_p \le\; & 4\int d\mathbf{r}|G_\ell(\mathbf{r})|\;\|f'_\delta(\mathbf{x})\|_r\|g'_\delta(\mathbf{x})\|_s \\ &+\; 4\int d\mathbf{r}|G_\ell(\mathbf{r})|\;\|f'_\delta(\mathbf{x})\|_r\int d\mathbf{r}|G_\ell(\mathbf{r})|\;\|g'_\delta(\mathbf{x})\|_s \\ =\; & O\left(\delta^{\sigma^f_r+\sigma^g_s}\right). \end{aligned} \tag{51}$$

Similarly, using (46),(48) we can show that non-local ultraviolet contributions to the gradient of a second-order central moment are bounded. For $1/p = 1/r + 1/s$, we have

$$\begin{aligned}
\|\boldsymbol{\nabla}\overline{\tau}_\ell(f'_\delta, g'_\delta)\|_p \leq & \quad \frac{4}{\ell}\Big\{ \int d\mathbf{r}|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_r \|g'_\delta(\mathbf{x})\|_s \\
& + \int d\mathbf{r}|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|g'_\delta(\mathbf{x})\|_s \\
& + \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_r \int d\mathbf{r}|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})| \; \|g'_\delta(\mathbf{x})\|_s \Big\} \\
= & \quad O\left(\frac{\delta^{\sigma_r^f + \sigma_s^g}}{\ell}\right). \qquad (52)
\end{aligned}$$

The non-local ultraviolet contributions to a third-order central moment of fields $f(\mathbf{x})$, $g(\mathbf{x})$, and $h(\mathbf{x})$ can be bounded using (46),(49). For $1/p = 1/r + 1/s + 1/t$, we have

$$\begin{aligned}
\|\overline{\tau}_\ell(f'_\delta, g'_\delta, h'_\delta)\|_p \leq & \; 8 \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_r \|g'_\delta(\mathbf{x})\|_s \|h'_\delta(\mathbf{x})\|_t \\
& + 8 \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|g'_\delta(\mathbf{x})\|_s \|h'_\delta(\mathbf{x})\|_t \\
& + 8 \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|g'_\delta(\mathbf{x})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_s \|h'_\delta(\mathbf{x})\|_t \\
& + 8 \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|h'_\delta(\mathbf{x})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_s \|g'_\delta(\mathbf{x})\|_t \\
& + 16 \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|f'_\delta(\mathbf{x})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|g'_\delta(\mathbf{x})\|_s \int d\mathbf{r}|G_\ell(\mathbf{r})| \; \|h'_\delta(\mathbf{x})\|_t \\
= & \; O\left(\delta^{\sigma_r^f + \sigma_s^g + \sigma_t^h}\right). \qquad (53)
\end{aligned}$$

B.2 Infrared locality

Infrared locality means that contributions to the energy flux across ℓ from scales $\Delta \gg \ell$ decay at least as fast as $\Delta^{-\alpha}$, for some $\alpha > 0$. We will now show that each of the factors in SGS flux terms (13) and (29), except for the density, is infrared local. As we have discussed above, it is physically expected that the kinetic energy cascade will depend on density fluctuations at the largest scales.

Non-local infrared contributions to a large-scale gradient of field $f(\mathbf{x})$ can be bounded using (45) as was shown by [19]:

$$\|\overline{(\boldsymbol{\nabla}\overline{f}_\Delta)}_\ell\|_p \leq \int d\mathbf{r}' \, |G_\ell(\mathbf{r}')| \, \frac{1}{\Delta} \int d\mathbf{r} \, |(\boldsymbol{\nabla}G)_\Delta(\mathbf{r})| \; \|\delta f(\mathbf{x};\mathbf{r})\|_p = O\left(\Delta^{\sigma_p^f - 1}\right). \qquad (54)$$

Non-local infrared contributions to a second-order central moment of fields $f(\mathbf{x})$ and $g(\mathbf{x})$ can be bounded

using (47) as was shown by [19]. For $1/p = 1/r + 1/s$, we have

$$\begin{aligned}
\|\overline{\tau}_\ell(\overline{f}_\Delta, \overline{g}_\Delta)\|_p \leq & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\overline{f}_\Delta(\mathbf{x};\mathbf{r})\|_r\|\delta\overline{g}_\Delta(\mathbf{x};\mathbf{r})\|_s \\
+ & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\overline{f}_\Delta(\mathbf{x};\mathbf{r})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\overline{g}_\Delta(\mathbf{x};\mathbf{r})\|_s \\
= & O\left(\Delta^{\sigma_r^f + \sigma_s^g - 2}\ \ell^2\right),
\end{aligned} \tag{55}$$

where the last step follows from $\|\delta\overline{g}_\Delta(\mathbf{x};\mathbf{r})\|_s = O(r\Delta^{\sigma_s^g - 1})$ (see [19]).

Using (48), we can also show that non-local infrared contributions to the gradient of a second-order central moment are bounded. Such a term only appears in the form $\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\tau}_\ell(\rho,\mathbf{u})$ in (29). For $1/p = 1/r + 1/s$, we have

$$\begin{aligned}
\|\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\tau}_\ell(\rho, \overline{\mathbf{u}}_\Delta)\|_p \leq & \frac{1}{\ell}\Bigg\{ \int d\mathbf{r}|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})|\boldsymbol{\cdot}\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_r\|\delta\overline{\mathbf{u}}_\Delta(\mathbf{x};\mathbf{r})\|_s \\
+ & \int d\mathbf{r}|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_r\boldsymbol{\cdot} \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\overline{\mathbf{u}}_\Delta(\mathbf{x};\mathbf{r})\|_s \\
+ & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_r \int d\mathbf{r}|(\boldsymbol{\nabla}G)_\ell(\mathbf{r})|\boldsymbol{\cdot}\|\delta\overline{\mathbf{u}}_\Delta(\mathbf{x};\mathbf{r})\|_s\Bigg\} \\
= & O\left(\Delta^{\sigma_s^u - 1}\ \ell^{\sigma_r^\rho}\right).
\end{aligned} \tag{56}$$

We can also show infrared locality of a third-order central moment. Such a term appears in deformation work (29) in the form $\overline{\tau}_\ell(\rho,\mathbf{u},\mathbf{u})$. Non-local infrared velocity contributions can be bounded using (49). For $1/p = 1/r + 1/s + 1/t$, we have

$$\begin{aligned}
\|\overline{\tau}_\ell(\rho, \overline{\mathbf{u}}_\Delta, \overline{\mathbf{u}}_\Delta)\|_p \leq & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_r\|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_s\|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_t \\
+ & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_s\|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_t \\
+ & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_s\|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_t \\
+ & \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_s\|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_t \\
+ & 2\int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\rho(\mathbf{x};\mathbf{r})\|_r \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_s \int d\mathbf{r}|G_\ell(\mathbf{r})|\ \|\delta\mathbf{u}_\Delta(\mathbf{x};\mathbf{r})\|_t \\
= & O\left(\Delta^{\sigma_s^u + \sigma_t^u - 2}\ \ell^{\sigma_r^\rho + 2}\right).
\end{aligned} \tag{57}$$

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